AD-783 038

GEOMETRIC-OPTICS THEORY FOR COHERENT SCATTERING OF MICROWAVES FROM THE OCEAN SURFACE

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Washington, D. C.

18 June 1974

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
NRL Report 7705

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER
AD-783 038

4. TITLE (and Subtitle)
GEOMETRIC-OPTICS THEORY FOR COHERENT SCATTERING OF MICROWAVES FROM THE OCEAN SURFACE

5. TYPE OF REPORT & PERIOD COVERED
A final report on an NRL problem.

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
Ronald M. Brown and Allen R. Miller

8. CONTRACT OR GRANT NUMBER(s)

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Naval Research Laboratory
Washington, D.C. 20375

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
NRL Problem R06-07B
RF 12-151-401:4555

11. CONTROLLING OFFICE NAME AND ADDRESS
Department of the Navy
Office of Naval Research
Arlington, Va. 22217

12. REPORT DATE
June 18, 1974

13. NUMBER OF PAGES
22

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)
Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)
Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
Wave scattering
Microwave scattering
Acoustic-wave scattering
Ocean-surface scattering
Forward scattering
Geometric optics

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In 1961 C. I. Beard found that experimental values of the rough-surface reflection coefficient for the coherent reflected field are larger than the values given by the generally accepted theoretical formula $\mathcal{R}_A = \exp[-2(2\pi g)^2]$ for values of $g = (\sigma_h \sin\psi)/\lambda$ greater 0.1 radian. Here σ_h is the standard deviation of the sea surface elevation, ψ is the grazing angle, and λ is the electromagnetic wavelength.

The expression for $\mathcal{R}_A$ is the Gaussian theoretical curve (sometimes called the "roughness factor") first published by W. S. Ament in 1953. Since then Ament's result

(Continued)

20. Abstract Continued

has been obtained by numerous other workers.

The disagreement between theory and experiment has been unresolved for over a decade; and it has been agreed generally that theoretical models based on geometric optics could not yield a result more powerful than that of Ament.

In this report it is shown that the coherent reflected field is given by $\mathfrak{R} = \exp[-2(2\pi g)^2] I_0[2(2\pi g)^2]$ for $0 < g < 0.3$, where $I_0(x)$ is the modified Bessel function $J_0(ix)$. This result is derived using geometric optics, assuming a spherical wavefront incident on a Gaussian collection of sinusoidal surface waves with uniform phase distribution. Further, $\mathfrak{R}$ agrees with Beard's experimental curve, with a systematic difference in g of 10%. Beard has estimated that wave elevations were within 10% of their correct values.

It is shown how $\mathfrak{R}_A$ is obtained, assuming a plane wavefront incident on a Gaussian collection of horizontal strips or equivalently simple functons.

CONTENTS

GEOMETRIC-OPTICS THEORY FOR COHERENT SCATTERING OF MICROWAVES FROM THE OCEAN SURFACE

INTRODUCTION

In 1961 C. I. Beard (1) found that experimental values of the coherent reflected field $|\bar{E}/E_\delta \Gamma|$ are larger than the values given by the generally accepted theoretical formula

$$\left|\frac{\bar{E}}{E_\delta \Gamma}\right|_{\mathcal{S}_A} = \exp\left[-2(2\pi g)^2\right]$$

for values of $g = (\sigma_h \sin \psi)/\lambda$ greater than 0.1 radian. Here $\bar{E}$ is the average electric field due to the "sea surface" $\mathcal{S}_A$, E_δ is the field due to the direct wave, Γ is the smooth sea reflection coefficient, σ_h is the standard deviation of the sea-surface elevation, ψ is the grazing angle, and λ is the electromagnetic wavelength (Fig. 1).

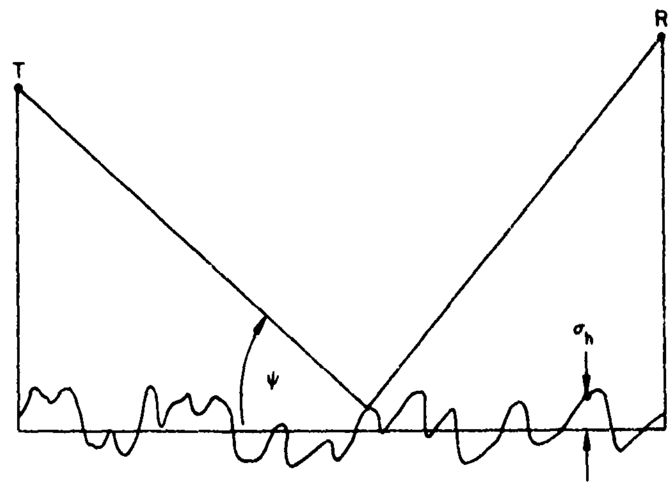

Fig. 1—Ray reflection off of a random sea surface

The expression $|\bar{E}/E_\delta \Gamma|\mathcal{S}_A$ is the Gaussian theoretical curve (sometimes called the "roughness factor") first published by W. S. Ament (2) in 1953. (Ament claims that the result was derived by Pekeris and, independently, by MacFarlane in the 40's.) Since then Ament's result has been obtained by numerous other workers. (See for example Ref. 3.)

Manuscript submitted December 14, 1973.

The disagreement between theory and experiment has been unresolved for over a decade; and it has been agreed generally that theoretical models based on geometric optics could not yield a result more powerful than that of Ament.

In this report we show that the coherent reflected field is given by

$$\left|\frac{\bar{E}}{E_\delta \Gamma}\right|_{\mathcal{S}} = \exp\left[-2(2\pi g)^2\right] I_0\left[2(2\pi g)^2\right]$$

for $0 \leq g \leq 0.3$, where $I_0(x)$ is the modified Bessel function $J_0(ix)$. We derive this result using geometric optics, assuming a spherical wavefront incident on a Gaussian collection $\mathcal{S}$ of sinusoidal surface waves. Further, $|\bar{E}/E_\delta \Gamma|\mathcal{S}$ agrees with Beard's experimental curve (Fig. 2), with a systematic difference in g of 10%. Beard has estimated that wave elevations were within 10% of their correct values.

We show how $|\bar{E}/E_\delta \Gamma|\mathcal{S}_A$ is obtained, assuming a plane wave front incident on a Gaussian collection $\mathcal{S}_A$ of horizontal strips.

BASIC ASSUMPTIONS AND DEFINITIONS

Suppose that T is an isotropic emitter of electromagnetic waves with wavelength λ situated at a distance h_2 above the mean sea surface and that R is an isotropic receiver which is at a distance h_1 above the mean sea surface. We take the mean sea height to be equal to zero. Suppose that the horizontal distance between T and R is d, where d is small enough that the earth's curvature need not be considered (Fig. 3). We assume

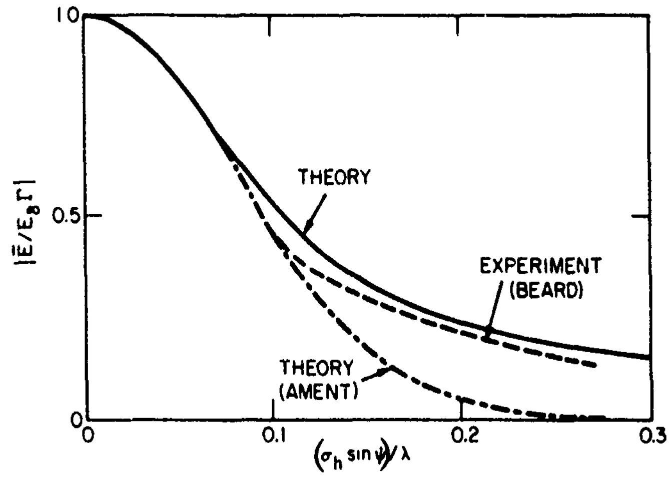

Fig. 2—Comparison of theoretical and experimental results for coherent forward scattering

that the sea surface can be described by some collection of functions $\mathcal{S}$. To be more specific, we assume that at any given moment the functions which we need to describe

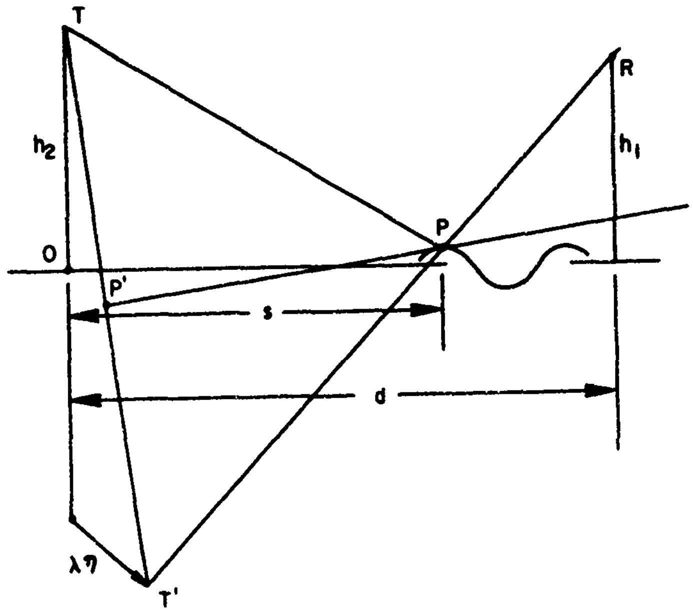

Fig. 3—Geometry of ray reflection

the sea surface locally are in $\mathcal{S}$. Let T' be the image of T with respect to the tangent line through P, a typical reflection point on the sea surface. Let $(\lambda\eta_1, \lambda\eta_2 - h_2)$ be the coordinates of T', so that the coordinates of P' are $(\lambda\eta_1/2, \lambda\eta_2/2)$. Note that when P is on the s axis and we have specular reflection, $\eta_1 = \eta_2 = 0$, i.e., the image of T through the s axis is at $T'_s(0, -h_2)$. When we have this situation, let ψ be the angle TPO. It is easy to see that $d \tan \psi = h_1 + h_2$. We remark that we will be using two independent coordinate systems, viz., the s, y system and the η_1, η_2 system with origin T'_s. Note that the coordinates of O in the s, y system are $(0, h_2)$ in the η_1, η_2 system.

In this work we use geometric optics, that is, we assume that only waves reflected from a favorable slope will reach the receiver and that the local angle of incidence will equal the local angle of reflection. Further, we assume that all reflections occur in the s, y plane. We also will assume that the sea surface is "slightly rough," so that over a long period of time the collection of points T', the image of $T \equiv \mathcal{J}_T$, will form a closed connected set which has no holes, that is, a closed, simply connected region. It will also be assumed that T'_s is an interior point of $\mathcal{J}_T$.

The following definitions will be helpful:

$$\xi_1 = h_1 + h_2 - \lambda\eta_2$$

$$\xi_2 = 2h_2 - \lambda\eta_2$$

$$\xi_3 = d - \lambda\eta_1$$

$$r_s^2 = d^2 + (h_1 + h_2)^2$$

$$\gamma_0 = \frac{2\pi r_s}{\lambda}$$

$$\gamma_1 = -2\pi \cos \psi$$

$$\gamma_2 = -2\pi \sin \psi$$

Note that r_s is the specular distance RT_s'. The distance RT' in general is $r = (\xi_1^2 + \xi_3^2)^{1/2}$.

We will show later that under suitable conditions T_s' is the center of symmetry of $\mathcal{J}_T$ and that if $(\lambda\eta_1, \lambda\eta_2) \in \mathcal{J}_T$, then

$$\left|\frac{\lambda\eta_1}{d}\right| \ll 1, \qquad \left|\frac{\lambda\eta_2}{h_2}\right| \ll 1,$$

so that $\mathcal{J}_T$ is bounded away from T and R. Let $\mathfrak{R}$ be the smallest rectangle which contains $\mathcal{J}_T$ and whose sides are parallel to the coordinate axes. Then in view of the previous inequalities, $\mathfrak{R}$ is a neighborhood of T_s', i.e., the dimensions of $\mathfrak{R}$ are small compared to d and h_2. We then are justified in expanding r in a Taylor series about $\eta_1 = \eta_2 = 0$ and dropping terms higher than those of first order in η_1 and η_2, giving

$$\frac{2\pi r}{\lambda} = \gamma_0 + \gamma_1\eta_1 + \gamma_2\eta_2 . \tag{1}$$

If we define the row vector $\gamma = [\gamma_1, \gamma_2]$ and let $\eta = \begin{bmatrix}\eta_1\\ \eta_2\end{bmatrix}$, we can write Eq. (1) as

$$\frac{2\pi r}{\lambda} = \gamma_0 + \gamma\eta . \tag{2}$$

CALCULATION OF THE ELECTRIC FIELD DUE TO A SINGLE REFLECTED RAY

The electric field E due to a single reflected ray is given by

$$E = E_\delta \frac{\delta}{r} |\Gamma| \exp\left(\frac{2\pi\delta}{\lambda} i\right) \exp\left[-\left(\frac{2\pi r}{\lambda} + \arg \Gamma\right) i\right], \tag{3}$$

where Γ is the complex reflection coefficient and E_δ is the electric field due to the direct wave which travels path length δ given by

$$\delta^2 = d^2 + (h_2 - h_1)^2 .$$

We consider now the quantity δ/r and state the following.

Observation: *If* $2h_2/d \lesssim 1$ ($\lesssim$ *means less than or of the order of*),

and

$$\frac{h_1}{d} \ll 1\,,$$

then

$$\frac{\delta}{r} \approx 1\,.$$

Proof: Using Eq. (1),

$$\frac{\delta}{r} = \frac{[d^2 + (h_2 - h_1)^2]^{1/2}}{[d^2 + (h_2 + h_1)^2]^{1/2} - \lambda\eta_1 \cos\psi - \lambda\eta_2 \sin\psi}$$

$$= \frac{\left[\dfrac{d^2 + (h_2 - h_1)^2}{d^2 + (h_2 + h_1)^2}\right]^{1/2}}{1 - \dfrac{\lambda\eta_1}{[d^2 + (h_2 + h_1)^2]^{1/2}} \cos\psi - \dfrac{\lambda\eta_2}{[d^2 + (h_2 + h_1)^2]^{1/2}} \sin\psi}\,.$$

Now

$$\left|\frac{\lambda\eta_1}{[d^2 + (h_2 + h_1)^2]^{1/2}}\right| < \left|\frac{\lambda\eta_1}{d}\right| \ll 1$$

and

$$\left|\frac{\lambda\eta_2}{[d^2 + (h_2 + h_1)^2]^{1/2}}\right| < \left|\frac{\lambda\eta_2}{h_2 + h_1}\right| < \left|\frac{\lambda\eta_2}{h_2}\right| \ll 1\,;$$

therefore

$$\frac{\delta^2}{r^2} \approx \frac{d^2 + h_1^2 + h_2^2 - 2h_1h_2}{d^2 + h_1^2 + h_2^2 + 2h_1h_2}\,.$$

However, $2h_2/d \lesssim 1$ and $h_1/d \ll 1$ imply that $2h_1h_2/d^2 \ll 1$, so that $\delta/r \approx 1$.

We then are justified in writing Eq. (3) approximately as

$$E = E_\delta\,|\Gamma| \exp\left(\frac{2\pi\delta}{\lambda}\,i\right) \exp\left[-\left(\frac{2\pi r}{\lambda} + \arg\Gamma\right) i\right]\,. \tag{4}$$

It is not hard to see that $\Gamma = \Gamma(\eta_1, \eta_2)$. Since $\mathcal{I}_T \subset \mathfrak{R}$ and $\mathfrak{R}$ is small, we will make the simplifying assumption that $\Gamma = \Gamma(0, 0) = \Gamma(\psi)$ in $\mathcal{I}_T$. For the surfaces we consider later the point (0, 0) will be the center of symmetry of $\mathcal{I}_T$, so that $\Gamma = \Gamma(\psi)$ can be viewed as some average in $\mathcal{I}_T$. Combining Eqs. (2) and (3) and defining

$$E_0 = E_\delta |\Gamma(\psi)| \exp\left(\frac{2\pi\delta}{\lambda} i\right) \exp(-\gamma_0 i) \exp\left[-i \arg \Gamma(\psi)\right]$$

gives

$$E = E_0 \exp(-i\gamma\eta). \tag{5}$$

We remark that $2h_2/d \lesssim 1$ and $h_1/d \ll 1$ imply that $(h_1 + h_2)/d \lesssim 1/2$, i.e., $\tan \psi \lesssim 1/2$, so that $\psi \lesssim 27°$. Hence Eq. (5) is valid provided that

$$\left|\frac{\lambda\eta_1}{d}\right| \ll 1,$$

$$\left|\frac{\lambda\eta_2}{h_2}\right| \ll 1,$$

and

$$\psi \lesssim 27°.$$

A RESULT DUE TO AMENT

Suppose $\mathcal{S}_A$ is the class of functions $y = H$, where the sea elevation H is distributed normally with standard deviation σ_H and $|2H| \ll \min\{h_1, h_2\}$. This is equivalent to the class of functions Ament used in Ref. 2, which is the class $H(s)$ where H "varies so slowly with s that, in the neighborhood of each s, the surface is approximately a plane parallel to the s axis."* It is easy to see that for the class $\mathcal{S}_A$,

*An equivalent way of defining the class $\mathcal{S}_A$ is the following:
Let $\{I_j\}$ be a disjoint collection of open intervals such that

$$\bigcup_{j=1}^{\infty} I_j = \mathfrak{R} - S,$$

where S is a set of measure zero and $\mathfrak{R}$ is the real line. Define

$$K_j(\chi) = \begin{cases} 1 & \text{if } \chi \in I_j \\ 0 & \text{if } \chi \notin I_j \end{cases}$$

Let $\{\alpha_j\}$ be a sequence of real numbers, and define a simple function

$$f(\chi) = \sum_{j=1}^{\infty} \alpha_j K_j(\chi).$$

Let $\mathcal{S}_A$ be the class of all simple functions (see Fig. 4).

$$\mathcal{J}_T = \left\{(\lambda\eta_1, \lambda\eta_2) \mid \eta_1 = 0, \eta_2 = \frac{2H}{\lambda}\right\},$$

so that in this case $\mathfrak{R} = \mathcal{J}_T$ a degenerate rectangle. Hence $\gamma\eta = \gamma_2\eta_2 = -4\pi H (\sin\psi)/\lambda$, so that from Eq. (5) we have

$$E(H) = E_0 \exp\left(\frac{4\pi i H \sin\psi}{\lambda}\right)$$
$$= E_0 \cos\left(\frac{4\pi H \sin\psi}{\lambda}\right) + i \sin\left(\frac{4\pi H \sin\psi}{\lambda}\right). \tag{6}$$

The average electric field $\overline{E}$ due to the class $\mathcal{S}_A$ is given by

$$\overline{E} = \frac{\int_{-H_0}^{H_0} E(H) \exp\left(-\frac{1}{2}\frac{H^2}{\sigma_H^2}\right) dH}{\int_{-H_0}^{H_0} \exp\left(-\frac{1}{2}\frac{H^2}{\sigma_H^2}\right) dH}.$$

If we substitute Eq. (6) into this equation, it is easy to see that we get

$$\overline{E} = E_0 \frac{\int_{-H_0}^{H_0} \exp\left(-\frac{1}{2}\frac{H^2}{\sigma_H^2}\right) \cos\left(\frac{4\pi H \sin\psi}{\lambda}\right) dH}{\int_{-H_0}^{H_0} \exp\left(-\frac{1}{2}\frac{H^2}{\sigma_H^2}\right) dH}.$$

To evaluate these integrals, we will let $H_0 \to +\infty$. The error we make will not be large, since the cosine function is bounded and the distribution tends to zero quickly when σ_H is small. Performing the required integrations and noting that $|E_0| = |E_\delta \Gamma(\psi)|$ gives

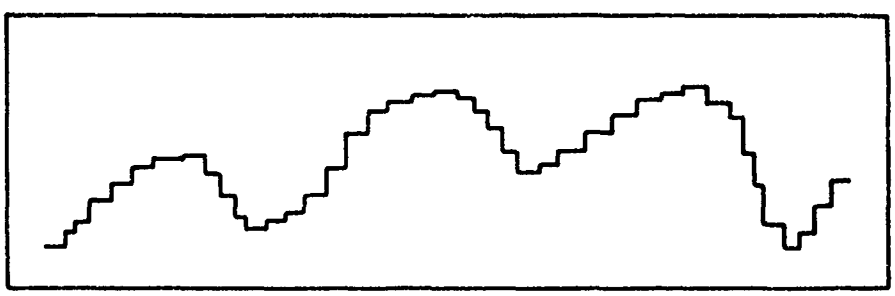

Fig. 4 — Typical simple function

$$\left|\frac{\bar{E}}{E_{\delta}\Gamma}\right|_{\mathcal{S}_A} = \exp\left[-2\left(\frac{2\pi\sigma_H \sin\psi}{\lambda}\right)^2\right] \tag{7}$$

Ament derived this result for a plane incident wave and the class $\mathcal{S}_A$. We have made the assumption that the incident wave front is spherical. However, if we are considering a plane wave, then $\Gamma \equiv \Gamma(\psi)$ in $\mathcal{J}_T$, and we have Ament's result.

CALCULATION OF THE IMAGE DUE TO THE CLASS $\mathcal{S}$

We assume now that the sea surface can be described by a Gaussian collection $\mathcal{S}$ of sinusoidal waves $y = H \sin[(2\pi/\mu)(s + \tilde{s})]$, where the sea-wave peak height $H \geqslant 0$ is normally distributed, μ is the sea-wave wavelength, and $\tilde{s}$ is the sea-wave phase, which is distributed uniformly.

Let $\beta = (2\pi/\mu)(s + \tilde{s})$, and note (Fig. 5) that the coordinates of P are $(s, H \sin\beta)$. The image $\mathcal{J}_T$ is determined by the following three equations:

$$\xi_1(d - s) = \xi_3(h_1 - H\sin\beta); \tag{8}$$

$$2\pi H\xi_2 \cos\beta = \mu\lambda\eta_1 ; \tag{9}$$

$$2\pi H(\lambda\eta_1 - 2s)\cos\beta = \mu(\lambda\eta_2 - 2H\sin\beta). \tag{10}$$

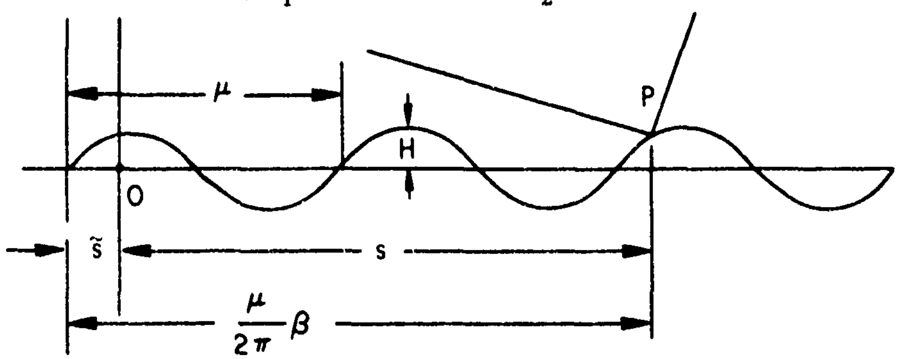

Fig. 5—Individual sinusoidal wave showing amplitude and phase

Equation (8) is the condition that T', P, and R lie in a straight line; Eq. (9) is the condition that the line through P' and P is perpendicular to the line through T' and T; and Eq. (10) is the condition that the slope of $y = H \sin\beta$ at P is equal to the slope of the line through P' and P.

We now state the following observation.

Observation: If

$$\frac{4\pi sH}{h_2\mu} \ll 1, \quad \frac{2H}{h_2} \ll 1, \quad \frac{2h_2}{d} \lesssim 1, \quad \frac{h_1 + h_2}{d} \lesssim 1, \tag{11}$$

then $\mathcal{I}_T$ *is approximately an ellipse and*

$$\partial\mathcal{I}_T = \left\{(\lambda\eta_1, \lambda\eta_2) | \eta_1 = \frac{2H}{\lambda}\frac{2\pi h_2}{\mu}\cos\beta, \eta_2 = \frac{2H}{\lambda}\left(\sin\beta - \frac{2\pi h_2}{\mu}\cos\psi\cos\beta\right)\right\},$$

where $\beta \in [0, 2\pi]$. *(∂ means "the boundary of.")*

We remark that the relations (11) aıe equivalent to the following:

$$2\pi\frac{H}{\mu} \ll 1, \quad \frac{2h_2}{d} \lesssim 1, \quad \frac{h_1 + h_2}{d} \lesssim 1, \tag{12a}$$

$$K \equiv \left|2\left(2\pi\frac{H}{\mu}\cos\beta\right)^2 - \frac{4\pi sH}{h_2\mu}\cos\beta + \frac{2H}{h_2}\sin\beta\right| \ll 1, \quad \beta \in [0, 2\pi]. \tag{12b}$$

The relations of (11) imply those of (12) since $\exists s \ni s/h_2 \gtrsim 1$ (for example, when $H = 0$, $s/h_2 = d/(h_1 + h_2) \gtrsim 1$, so that certainly $\sqrt{2}s/h_2 \gtrsim 1$, and $2s/h_2 \gtrsim 1$; thus

$$\sqrt{2}\left(2\pi\frac{H}{\mu}\right) \ll 1, \quad 2\pi\frac{H}{\mu} \ll 1, \quad K \ll 1.$$

Relations (12) imply (11), for when $\beta = \pi/2$, we have $2H/h_2 \ll 1$.

When $\beta = 0$,

$$K = \left|2\left(2\pi\frac{H}{\mu}\right)^2 - \frac{4\pi sH}{h_2\mu}\right| \ll 1.$$

Hence we must have

$$2\left(2\pi\frac{H}{\mu}\right)^2 \ll 1, \quad \frac{4\pi sH}{h_2\mu} \ll 1$$

or

$$2\left(2\pi\frac{H}{\mu}\right)^2 \sim \frac{4\pi sH}{h_2\mu},$$

which implies that

$$2\pi\frac{H}{\mu} \sim \frac{s}{h_2}.$$

But

$$2\pi \frac{H}{\mu} \ll 1 \,,$$

and so

$$\frac{s}{h_2} \ll 1 \,,$$

a result which is not always true. Hence

$$\frac{4\pi s H}{h_2 \mu} \ll 1 \,.$$

Further, if we eliminate the parameter β, we have

$$\partial \mathcal{G}_T = \left\{ (\lambda\eta_1, \lambda\eta_2) \middle| \left(\frac{2H}{\lambda}\right)^2 = \eta' \Delta \eta \right\} ,$$

where

$$\Delta \equiv \begin{bmatrix} \left(\frac{\mu}{2\pi h_2}\right)^2 + \cot^2 \psi & \cot \psi \\ \cot \psi & 1 \end{bmatrix} .$$

Proof: From Eqs. (9) and (10) respectively,

$$2\pi \frac{H}{\mu} \cos\beta = \frac{\lambda\eta_1}{h_2} \frac{1}{\left(2 - \frac{\lambda\eta_2}{h_2}\right)} \,, \tag{13}$$

and

$$2\pi \frac{H}{\mu} \cos\beta = \frac{\frac{\lambda\eta_2}{h_2} - \frac{2H}{h_2} \sin\beta}{\frac{\lambda\eta_1}{h_2} - \frac{2s}{h_2}} .$$

Eliminating $\lambda\eta_1/h_2$ from these equations gives

$$2\pi \frac{H}{\mu} \cos\beta = \frac{\frac{\lambda\eta_2}{h_2} - \frac{2H}{h_2} \sin\beta}{2\pi \frac{H}{\mu}\left(2 - \frac{\lambda\eta_2}{h_2}\right) \cos\beta - \frac{2s}{h_2}} .$$

Solving for $\lambda\eta_2/h_2$ gives

$$\frac{\lambda\eta_2}{h_2} = \frac{2\left(2\pi\frac{H}{\mu}\cos\beta\right)^2 - \frac{2s}{h_2}\left(2\pi\frac{H}{\mu}\cos\beta\right) + \frac{2H}{h_2}\sin\beta}{1 + \left(2\pi\frac{H}{\mu}\cos\beta\right)^2} .$$

Hence

$$\left[1 + \left(2\pi\frac{H}{\mu}\cos\beta\right)^2\right] \left|\frac{\lambda\eta_2}{h_2}\right| = K \ll 1 ,$$

and it is clear then that

$$\left|\frac{\lambda\eta_2}{h_2}\right| \ll 1 .$$

Now we can write Eq. (13) as

$$2\pi\frac{H}{\mu}\cos\beta = \frac{\lambda\eta_1}{2h_2}\,\frac{1}{1 - \frac{\lambda\eta_2}{2h_2}} ,$$

and since $|\lambda\eta_2/2h_2| \ll 1$, we have approximately

$$\eta_1 = \frac{4\pi h_2 H}{\lambda\mu}\cos\beta . \tag{14}$$

From Eq. (8) and (10) respectively,

$$\frac{\lambda\eta_2}{2} = H\sin\beta + 2\pi\frac{H}{\mu}\cos\beta\left(\frac{\lambda\eta_1}{2} - s\right) , \tag{15}$$

and

$$s = \frac{d - \lambda\eta_1}{h_1 + h_2 - \lambda\eta_2}(H\sin\beta - h_1) + d . \tag{16}$$

Eliminating $\lambda\eta_1$ from Eqs. (14) and (15) gives

$$\frac{\lambda\eta_2}{2} = H\sin\beta + \left(2\pi\frac{H}{\mu}\cos\beta\right)^2 h_2 - \left(2\pi\frac{H}{\mu}\cos\beta\right) s : \tag{17}$$

Equation (16) can be written as

$$s = \frac{d}{h_1 + h_2}(H \sin\beta - h_1)\left(1 - \frac{\lambda\eta_1}{d}\right)\left(\frac{1}{1 - \dfrac{\lambda\eta_2}{h_1 + h_2}}\right) + d\,. \tag{18}$$

Clearly

$$\left|\frac{\lambda\eta_2}{h_1 + h_2}\right| \ll 1\,,$$

and since from Eq. (14)

$$\left|\frac{\lambda\eta_1}{d}\right| = \left|2\pi\frac{H}{\mu}\,\frac{2h_2}{d}\,\cos\beta\right| \ll 1\,,$$

we can write Eq. (18) approximately as

$$\begin{aligned} s &= \frac{d}{h_1 + h_2}(H \sin\beta - h_1) + d \\ &= \cot\psi\,(h_2 + H\sin\beta)\,. \end{aligned} \tag{19}$$

From Eqs. (17) and (19), we have

$$\begin{aligned} \frac{\lambda\eta_2}{2} = H\sin\beta &+ \left(2\pi\frac{H}{\mu}\cos\beta\right)^2 h_2 - \left(2\pi\frac{H}{\mu}\cos\beta\right) h_2\cot\psi \\ &- \left(2\pi\frac{H}{\mu}\cos\beta\right) H\cot\psi\sin\beta\,. \end{aligned} \tag{20}$$

Since

$$\left|\frac{\left(2\pi\dfrac{H}{\mu}\cos\beta\right) H\cot\psi\sin\beta}{\left(2\pi\dfrac{H}{\mu}\cos\beta\right) h_2\cot\psi}\right| = \left|\frac{H\sin\beta}{h_2}\right| \ll 1$$

and

$$\left| \frac{\left(2\pi \frac{H}{\mu} \cos\beta\right)^2 h_2}{\left(2\pi \frac{H}{\mu} \cos\beta\right) h_2 \cot\psi} \right| = \left| 2\pi \frac{H}{\mu} \left(\frac{h_1 + h_2}{d}\right) \cos\beta \right|$$

$$\leqslant \left(2\pi \frac{H}{\mu}\right) \left(\frac{h_1 + h_2}{d}\right) \ll 1 ,$$

we can write Eq. (20) approximately as

$$\frac{\lambda\eta_2}{2} = H \sin\beta - \left(2\pi \frac{H}{\mu} \cos\beta\right) h_2 \cot\psi ,$$

and so

$$\eta_2 = \frac{2H}{\lambda} \left(\sin\beta - \frac{2\pi h_2}{\mu} \cot\psi \cos\beta\right) . \qquad (21)$$

We easily can verify that

$$\eta' \Delta\eta = \left(\frac{2H}{\lambda}\right)^2$$

and check that it is indeed an ellipse. The family of ellipses is shown in Fig. 6. If we let $\beta = \pi/2, 3\pi/2$, we obtain the interesting result that $\mathcal{I}_T$ (Class $\mathcal{S}$) $\cap$ {y axis} = $\mathcal{I}_T$ (Class $\mathcal{S}_A$), so that the major axis of the ellipse $(2H/\lambda)^2 = \eta'\Delta\eta$ must be greater than $4H/\lambda$.

AVERAGE ELECTRIC FIELD DUE TO THE CLASS $\mathcal{S}$

For a given H and β, the electric field due to a single reflected ray is given by Eq. (5): $E = E_0 \exp(-i\gamma\eta)$. We easily find that

$$\gamma\eta = \frac{4\pi H}{\lambda} \sin\psi \sin\beta ,$$

so that

$$E(H,\beta) = E_0 \exp\left(-\frac{4\pi i H}{\lambda} \sin\psi \sin\beta\right) . \qquad (22)$$

When $\beta = \pi/2, 3\pi/2$, we get Eq. (6). The physical interpretation of $\gamma\eta$ is shown in Fig. 7.

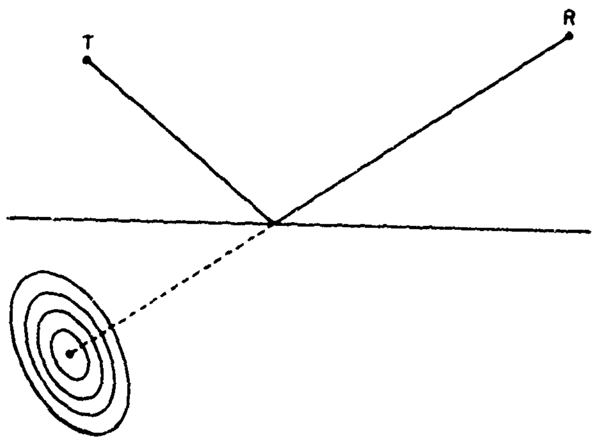

Fig. 6—Family of image ellipses

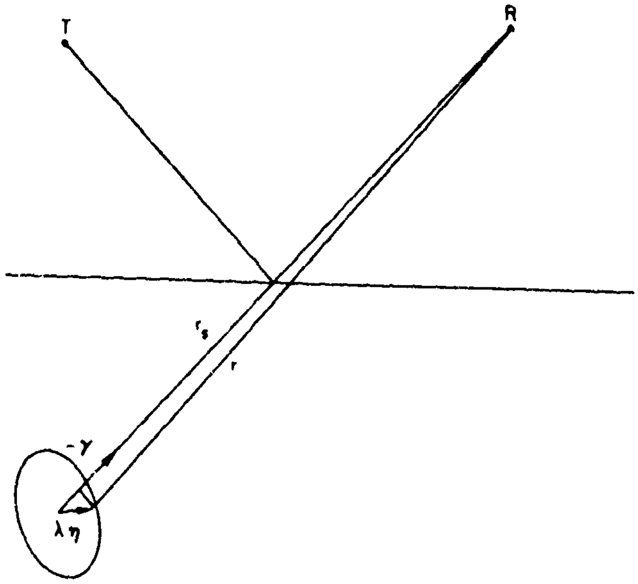

Fig. 7—Path length from an arbitrary point on an image ellipse to the receiver, which determines phase

We now define

$$\tilde{E}(H) \equiv \frac{\displaystyle\int_{\beta=0}^{\beta=2\pi} E(H, \beta)\, d\tilde{s}}{\displaystyle\int_{\beta=0}^{\beta=2\pi} d\tilde{s}} . \tag{23}$$

Equation (23) gives the average elective field due to all reflections such that T′ is in $\partial \mathfrak{J}_T$.

We recall that

$$\beta = \frac{2\pi}{\mu}(s + \tilde{s})$$

and

$$s = \cot \psi\, (h_2 + H \sin \beta),$$

so that

$$\frac{\mu}{2\pi}\beta = h_2 \cot \psi + H \cot \psi \sin \beta + \tilde{s} .$$

Taking differentials in this last equation gives

$$\frac{\mu}{2\pi} d\beta = H \cot \psi \cos \beta\, d\beta + d\tilde{s} ,$$

so that

$$d\tilde{s} = \left(\frac{\mu}{2\pi} - H \cot \psi \cos \beta\right) d\beta .$$

Hence

$$\tilde{E}(H) = \frac{\displaystyle\int_{\beta=0}^{\beta=2\pi} E(H, \beta) \left(\frac{\mu}{2\pi} - H \cot \psi \cos \beta\right) d\beta}{\displaystyle\int_{\beta=0}^{\beta=2\pi} \left(\frac{\mu}{2\pi} - H \cot \psi \cos \beta\right) d\beta}$$

$$= \frac{1}{2\pi} \int_{\beta=0}^{\beta=2\pi} E(H, \beta)\, d\beta - \frac{H \cot \psi}{\mu} \int_{\beta=0}^{\beta=2\pi} E(H, \beta) \cos \beta\, d\beta .$$

But $E(H, \beta)$ is a function of $\sin \beta$, so that

$$\int_{\beta=0}^{\beta=2\pi} E(H, \beta) \cos \beta \, d\beta = \int_{\beta=0}^{\beta=2\pi} E(H, \beta) \, d \sin \beta = 0 .$$

Hence

$$\tilde{E}(H) = \frac{1}{2\pi} \int_{\beta=0}^{\beta=2\pi} E(H, \beta) \, d\beta ,$$

and on performing the required integration, we obtain

$$\tilde{E}(H) = E_0 J_0\left(\frac{4\pi H}{\lambda} \sin \psi\right) , \tag{24}$$

where J_0 is the Bessel function of order 0. Note that μ does not appear in Eq. (24).

Now let $\overline{E}$ be the average electric field due to the Gaussian collection of all sinusoidal waves, i.e.,

$$\overline{E} = \frac{\displaystyle\int_{H=0}^{H=H_0} \tilde{E}(H) \exp\left(-\frac{1}{2}\frac{H^2}{\sigma_H^2}\right) dH}{\displaystyle\int_{H=0}^{H=H_0} \exp\left(-\frac{1}{2}\frac{H^2}{\sigma_H^2}\right) dH} , \tag{25}$$

where $2H_0 \ll \min \{h_1, h_2\}$ and σ_H is the standard deviation in the peak height H. Substituting Eq. (24) into Eq. (25) gives

$$\overline{E} = E_0 \frac{\displaystyle\int_{H=0}^{H=H_0} J_0\left(4\pi \frac{H}{\lambda} \sin \psi\right) \exp\left(-\frac{1}{2}\frac{H^2}{\sigma_H^2}\right) dH}{\displaystyle\int_{H=0}^{H=H_0} \exp\left(-\frac{1}{2}\frac{H^2}{\sigma_H^2}\right) dH} .$$

As before, we let $H_0 \to +\infty$ without making a large error, and on performing the required integrations (and noting that $|E_0| = |E_\delta \Gamma|$), we obtain

$$\left|\frac{\overline{E}}{E_\delta \Gamma}\right| = \exp\left[-\left(\frac{2\pi\sigma_H \sin \psi}{\lambda}\right)^2\right] I_0\left[\left(\frac{2\pi\sigma_H \sin \psi}{\lambda}\right)^2\right] ,$$

where $I_0(x)$ is the modified Bessel function $J_0(ix)$.

If σ_h is the standard deviation of the sea elevation, then, since we have considered sinusoidal waves, it is clear that $\sigma_H = \sqrt{2}\sigma_h$ (Fig. 8). Hence we can write

$$\left|\frac{\overline{E}}{E_\delta \Gamma}\right| = \exp\left[-2\left(\frac{2\pi\sigma_h \sin\psi}{\lambda}\right)^2\right] I_0\left[2\left(\frac{2\pi\sigma_h \sin\psi}{\lambda}\right)^2\right] .$$

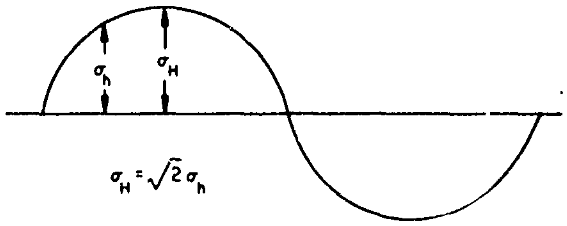

Fig. 8--Relationship between standard deviation in peak height, σ_H, and standard deviation in elevation, σ_h, for a sinusoidal wave

We remark that the use of sinusoidal surface waves is a mechanism for obtaining a rich source of wave-height wave-slope pairs. Further, it seems that we obtain enough of these pairs distributed in the right way to describe the sea surface over a long period of time.

To sum up we provide the following tabulation:

Table 1
Coherent Field for Various Sea-Surface/Incident-Wave Combinations

Sea Surface	*Incident Wave*	*Result*
Simple functions	Plane	Ament: $\lvert\overline{E}/E_\delta\Gamma\rvert_{\mathcal{S}_A}$
Simple functions	Spherical	Ament: $\lvert\overline{E}/E_\delta\Gamma\rvert_{\mathcal{S}_A}$
Sinusoidal waves	Plane	Ament-like: $\lvert\overline{E}/E_\delta\Gamma\rvert_{\mathcal{S}_{\hat{A}}}$
Sinusoidal waves	Spherical	$\lvert\overline{E}/E_\delta\Gamma\rvert_{\mathcal{S}}$

The Ament-like result is obtained by substituting $\sigma_H = \sqrt{2}\sigma_h$ in

$$\left|\frac{\overline{E}}{E_\delta \Gamma}\right|_{\mathcal{S}_A} = \exp\left[-2\left(\frac{2\pi\sigma_H \sin\psi}{\lambda}\right)^2\right] ,$$

giving

$$\left|\frac{\overline{E}}{E_\delta \Gamma}\right|_{\mathcal{S}_{\hat{A}}} = \exp\left[-4\left(\frac{2\pi\sigma_h \sin\psi}{\lambda}\right)^2\right] .$$

This is justified since when the incident wave is plane, the only reflections that can occur are off the peaks and troughs (assuming no shadowing) of the waves (Fig. 9).

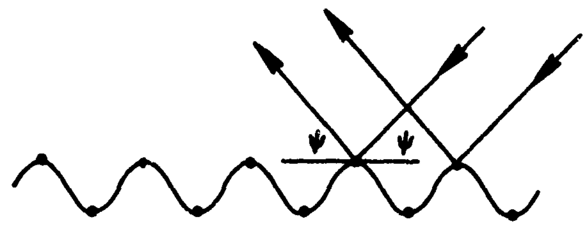

Fig. 9—Reflection points for a plan incident wave

A RESULT DUE TO BECKMANN

Since the completion of this work, it has been realized that another, but different, theoretical treatment based on geometric optics had been done previously by Beckmann (4)*. Beckmann's report was undertaken to try to calculate theoretically the effect of spherical-wavefront illumination for comparison with Beard's experimental results of the statistics of the incoherent field scattered from a random water surface (5). In the process of calculating the incoherent field, Beckmann also derived expressions for the coherent field. It seems that these were not pursued at the time since the interest was in the statistics of the incoherent field. Beckmann obtained

$$\left|\frac{\overline{E}}{E_\delta \Gamma}\right|_B = \exp(-K)\left[1 + \frac{K}{\pi}\,{}_1F_1{}^2\left(\frac{1}{2};\frac{3}{2};K\right)\right]^{1/2} ,$$

$$K = \left(10\,\frac{\sigma_h \sin\psi}{\lambda}\right)^2 ;$$

where ${}_1F_1(1/2; 3/2; K)$ is a confluent hypergeometric function. We have extracted this result from Ref. 4, Eqs. (49), (50), (51), and (85). Numerical computations indicate that

$$\left|\frac{\overline{E}}{E_\delta \Gamma}\right|_{S_A} < \left|\frac{\overline{E}}{E_\delta \Gamma}\right|_B < \left|\frac{\overline{E}}{E_\delta \Gamma}\right|_S \qquad \text{for } 0.1 \leqslant g \leqslant 0.3 .$$

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*The authors thank C. I. Beard for pointing out this reference.

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